Model Predictive Control

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Spring Semester 2014

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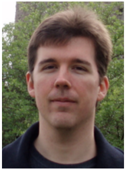

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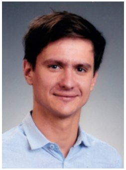

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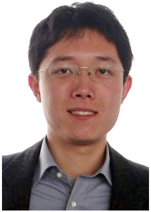

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Lecture Material

Compilation:	Xiaojing (George) Zhang, David Sturzenegger Please email suggestions and typos to xiaozhan@control.ee.ethz.ch sturzenegger@control.ee.ethz.ch
Software:	*Beamer* for LaTeX by Till Tantau & Vedran Milentić
Printed material:	Available in 2-page layout Sold during lecture or later at ETL I23 for CHF 30
Recordings:	Entire lecture is video recorded Link will be provided on lecture homepage
Homepage:	http://control.ee.ethz.ch/index.cgi?page=lectures

About the Lecture

Duration: Monday, 17. Feb 2014 – Friday, 28. Feb 2014

Credits: 6 credits for passing the exam

Exercises: Computer excercises, ETZ D61.1/2

Exam: Fri, 14. March 2014 (written), Location: tba

Week 1:

Date	Topic	Lectures		Exercises
		Time	Location	
Mon, Feb 17	Linear Systems I	9.15 – 12	HG E3	13.15 – 17
Tue, Feb 18	Linear Systems II	9.15 – 12	HG E3	13.15 – 17
Wed, Feb 19	Optimization I	9.15 – 12	HG D16.2	13.15 – 17
Thu, Feb 20	Optimization II	9.15 – 12	HG D16.2	13.15 – 17
Fri, Feb 21	Introduction to MPC	9.15 – 12	HG E3	13.15 – 17

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Week 2:

Date	Topic	Lectures		Exercises
		Time	Location	
Mon, Feb 24	Numerical Methods	9.15 – 12	HG E3	13.15 – 17
Tue, Feb 25	$\boxed{\text{Advanced Topics I}}$	9.15 – 12	HG E3	13.15 – 17
Wed, Feb 26	Invited Talks	9.15 – 17	HG D16.2	—
Thu, Feb 27	Design Exercise	—	—	10.15 – 17
Fri, Feb 28	$\boxed{\text{Advanced Topics II}}$	9.15 – 12	HG D16.2	—

Model Predictive Control
Part I – Introduction

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Spring Semester 2014

Literature

Model Predictive Control:

- Predictive Control for linear and hybrid systems, F. Borrelli, A. Bemporad, M. Morari, 2013 Cambridge University Press [http://www.mpc.berkeley.edu/mpc-course-material]
- Model Predictive Control: Theory and Design, James B. Rawlings and David Q. Mayne, 2009 Nob Hill Publishing
- Predictive Control with Constraints, Jan Maciejowski, 2000 Prentice Hall

Optimization:

- Convex Optimization, Stephen Boyd and Lieven Vandenberghe, 2004 Cambridge University Press
- Numerical Optimization, Jorge Nocedal and Stephen Wright, 2006 Springer

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Main Idea

Objective:

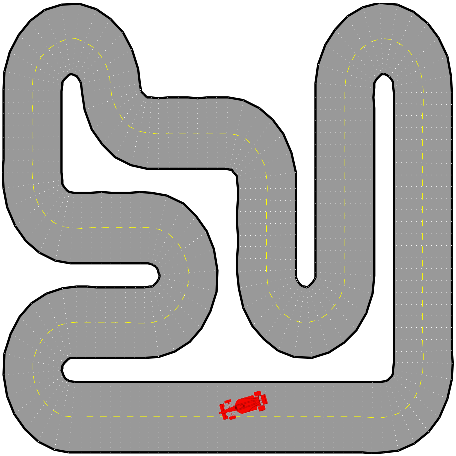
- Minimize lap time

Constraints:

- Avoid other cars
- Stay on road
- Don't skid
- Limited acceleration

Intuitive approach:

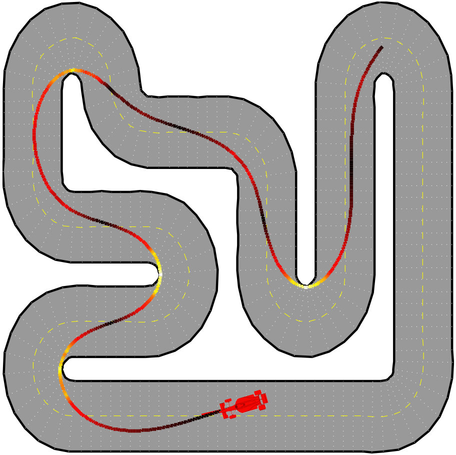
- Look forward and plan path based on
  - Road conditions
  - Upcoming corners
  - Abilities of car
  - etc...



Optimization-Based Control

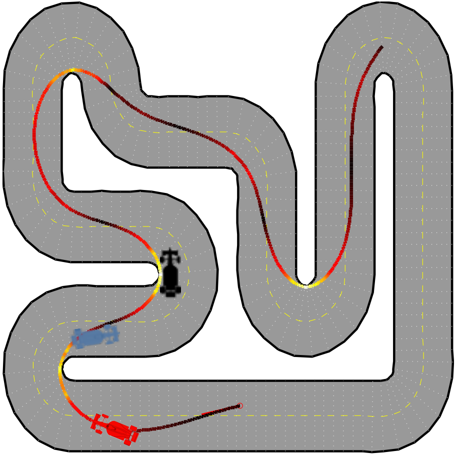
Minimize	(lap time)
while	avoid other cars
	stay on road
	…

- Solve **optimization problem** to compute minimum-time path



Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

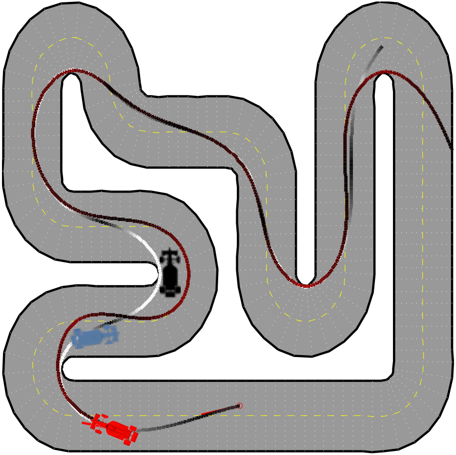


- Solve **optimization problem** to compute minimum-time path
- What to do if something unexpected happens?
  - We didn't see a car around the corner!
  - Must introduce *feedback*

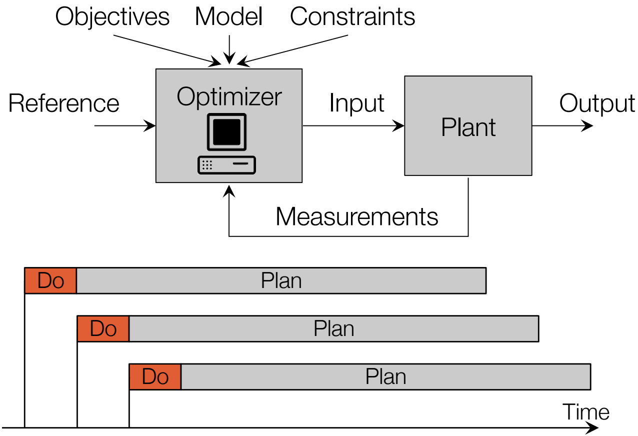
Optimization-Based Control

Minimize	(lap time)
while	avoid other cars
	stay on road
	...

- Solve **optimization problem** to compute minimum-time path
- Obtain series of planned control actions
- Apply *first* control action
- Repeat the planning procedure



Model Predictive Control



Receding horizon strategy introduces **feedback**.

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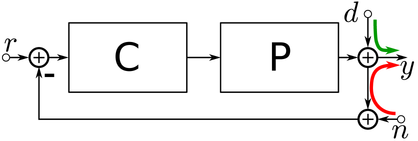
Two Different Perspectives

Classical design: design C

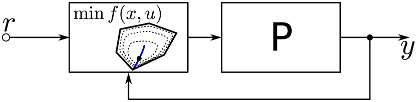


Dominant issues addressed

- Disturbance rejection $(d \to y)$
- Noise insensitivity $(n \to y)$
- Model uncertainty

(usually in *frequency domain*)

MPC: real-time, repeated optimization to choose $u(t)$



Dominant issues addressed

- Control constraints (limits)
- Process constraints (safety)

(usually in *time domain*)

Constraints in Control

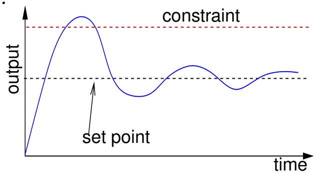
All physical systems have **constraints**:

- Physical constraints, e.g. actuator limits
- Performance constraints, e.g. overshoot
- Safety constraints, e.g. temperature/pressure limits

Optimal operating points are often near constraints.

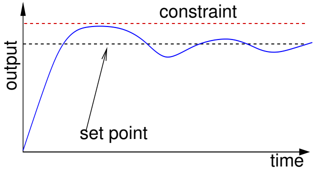
Classical control methods:

- Ad hoc constraint management
- Set point sufficiently far from constraints
- Suboptimal plant operation



Predictive control:

- Constraints included in the design
- Set point optimal
- Optimal plant operation



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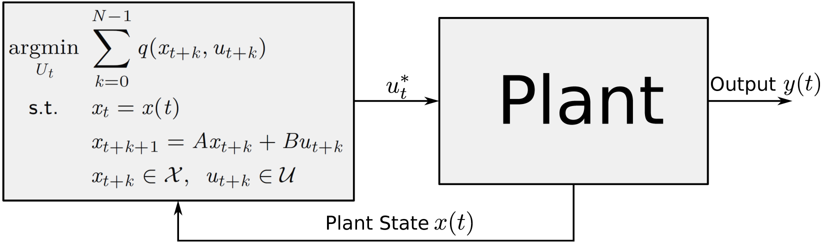
MPC: Mathematical Formulation

$$
\begin{aligned}
U_t^*(x(t)) := \operatorname*{argmin}_{U_t} \; & \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) && \\
\text{subj. to } \; & x_t = x(t) && \text{measurement} \\
& x_{t+k+1} = Ax_{t+k} + Bu_{t+k} && \text{system model} \\
& x_{t+k} \in \mathcal{X} && \text{state constraints} \\
& u_{t+k} \in \mathcal{U} && \text{input constraints} \\
& U_t = \{u_0, u_1, \ldots, u_{N-1}\} && \text{optimization variables}
\end{aligned}
$$

Problem is defined by

- **Objective** that is minimized,
  e.g., distance from origin, sum of squared/absolute errors, economic,...
- Internal **system model** to predict system behavior
  e.g., linear, nonlinear, single-/multi-variable, ...
- **Constraints** that have to be satisfied
  e.g., on inputs, outputs, states, linear, quadratic,...

MPC: Mathematical Formulation



At each sample time:

- Measure / estimate current state $x(t)$
- Find the optimal input sequence for the entire planning window N:
  $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the *first* control action u_t^*

Problem Formulation

Quadratic cost function

$$J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \tag{2}$$

with $P \succeq 0,\ Q \succeq 0,\ R \succ 0$.
Constrained Finite Time Optimal Control problem (CFTOC).

$$\boxed{\begin{array}{rll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array}} \tag{3}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Construction of the QP with substitution

- **Step 1**: Rewrite the cost as (see lectures on Day 1 & 2)

$$\begin{aligned} J_0(x(0), U_0) &= U_0' H U_0 + 2x(0)' F U_0 + x(0)' Y x(0) \\ &= [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \end{aligned}$$

  Note: $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$ since $J_0(x(0), U_0) \geq 0$ by assumption.

- **Step 2**: Rewrite the constraints compactly as (details provided on the next slide)

$$G_0 U_0 \leq w_0 + E_0 x(0)$$

- **Step 3**: Rewrite the optimal control problem as

$$\boxed{\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0) \end{aligned}}$$

Solution

$$
\begin{aligned}
J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\
\text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0)
\end{aligned}
$$

For a given x(0) U_0^* can be found via a QP solver.

Summary

- Obtain a model of the system
- Design a state observer
- Define optimal control problem
- Set up optimization problem in optimization software
- Solve optimization problem to get optimal control sequence
- Verify that closed-loop system performs as desired,
  e.g., check performance criteria, robustness, real-time aspects,...

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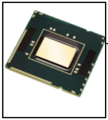
MPC: Applications



Computer control — ns

μs — Power systems

Traction control — ms

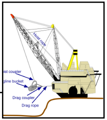
Seconds — Buildings

Refineries — Minutes

Hours — Nurse rostering

Train scheduling — Days

Weeks — Production planning



Important Aspects of Model Predictive Control

Main advantages:

- Systematic approach for handling *constraints*
- High *performance* controller

Main challenges:

- *Implementation*
  MPC problem has to be solved in real-time, i.e. within the sampling interval of the system, and with available hardware (storage, processor,...).
- *Stability*
  Closed-loop stability, i.e. convergence, is not automatically guaranteed
- *Robustness*
  The closed-loop system is not necessarily robust against uncertainties or disturbances
- *Feasibility*
  Optimization problem may become infeasible at some future time step, i.e. there may not exist a plan satisfying all constraints

Model Predictive Control
Part III – Feasibility and Stability

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Spring Semester 2014
revised 29.04.2014

Infinite Time Constrained Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^*(x(0)) = \min \sum_{k=0}^{\infty} q(x_k, u_k) \\
\text{s.t. } & x_{k+1} = Ax_k + Bu_k, k = 0, \dots, N-1 \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \dots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

- **Stage cost** $q(x, u)$ describes "cost" of being in state x and applying input u
- Optimizing over a trajectory provides a **tradeoff between short- and long-term benefits** of actions
- We'll see that such a control law has many beneficial properties...
  ... but we can't compute it: there are an **infinite number of variables**

Receding Horizon Control (what we can sometimes solve)

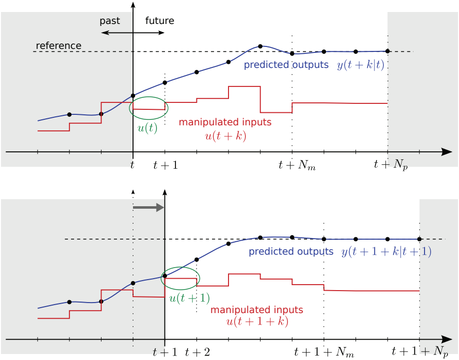
$$
\begin{array}{rll}
J_t^*(x(t)) = & \min\limits_{U_t} & p(x_{t+N}) + \sum\limits_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
& \text{subj. to} & x_{t+k+1} = Ax_{t+k} + Bu_{t+k},\ k = 0, \ldots, N-1 \\
& & x_{t+k} \in \mathcal{X},\ u_{t+k} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& & x_{t+N} \in \mathcal{X}_f \\
& & x_t = x(t)
\end{array} \quad (1)
$$

where $\mathcal{U}_t = \{u_t, \ldots, u_{t+N-1}\}$.

Truncate after a finite horizon:

- $p(x_{t+N})$: Approximates the 'tail' of the cost
- $\mathcal{X}_f$: Approximates the 'tail' of the constraints

On-line Receding Horizon Control



1. At each sampling time, solve a **CFTOC**.
2. Apply the optimal input **only during** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
4. The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

On-line Receding Horizon Control

1) MEASURE the state $x(t)$ at time instance t
2) OBTAIN $U_t^*(x(t))$ by solving the optimization problem in (1)
3) IF $U_t^*(x(t)) = \emptyset$ THEN 'problem infeasible' STOP
4) APPLY the first element u_t^* of U_t^* to the system
5) WAIT for the new sampling time $t+1$, GOTO 1)

Note that, we need a constrained optimization solver for step 2).

History of MPC

- **A. I. Propoi, 1963**, “Use of linear programming methods for synthesizing sampled-data automatic systems”, *Automation and Remote Control*.
- **J. Richalet et al., 1978** “Model predictive heuristic control- application to industrial processes”. *Automatica*, 14:413-428.
  - known as **IDCOM (Identification and Command)**
  - impulse response model for the plant, linear in inputs or internal variables (**only stable plants**)
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by a reference trajectory
  - **ad hoc** input and output constraints
  - optimal inputs computed using a heuristic iterative algorithm, interpreted as the dual of identification
  - controller was not a transfer function, hence called **heuristic**

History of MPC

- 1970s: Cutler suggested MPC in his PhD proposal at the University of Houston in 1969 and introduced it later at Shell under the name Dynamic Matrix Control. **C. R. Cutler, B. L. Ramaker, 1979** "Dynamic matrix control – a computer control algorithm". *AICHE National Meeting*, Houston, TX.
  - successful in the petro-chemical industry
  - linear step response model for the plant
  - quadratic performance objective over a finite prediction horizon
  - future plant output behavior specified by trying to follow the set-point as closely as possible
  - input and output constraints included in the formulation
  - optimal inputs computed as the solution to a least–squares problem
  - **ad hoc** input and output constraints. Additional equation added online to account for constraints. Hence a **dynamic matrix** in the least squares problem.
- **C. Cutler, A. Morshedi, J. Haydel, 1983**. "An industrial perspective on advanced control". *AICHE Annual Meeting*, Washington, DC.
  - Standard QP problem formulated in order to systematically account for constraints.

History of MPC

- Mid 1990s: extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability
- 2000s: development of tractable robust MPC approaches; nonlinear and hybrid MPC; MPC for very fast systems
- 2010s: stochastic MPC; distributed large-scale MPC; economic MPC

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MPC Features

Pros

- Any model
  - linear
  - nonlinear
  - single/multivariable
  - time delays
  - constraints
- Any objective:
  - sum of squared errors
  - sum of absolute errors (i.e., integral)
  - worst error over time
  - economic objective

Cons

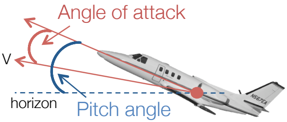
- Computationally demanding in the general case
- May or may not be stable
- May or may not be feasible

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0,\ 0,\ -1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_t^T Q x_t + u_k^T R u_k$$
$$\text{s.t. } x_{k+1} = A x_k + B u_k$$
$$x_0 = x(t)$$

MPC

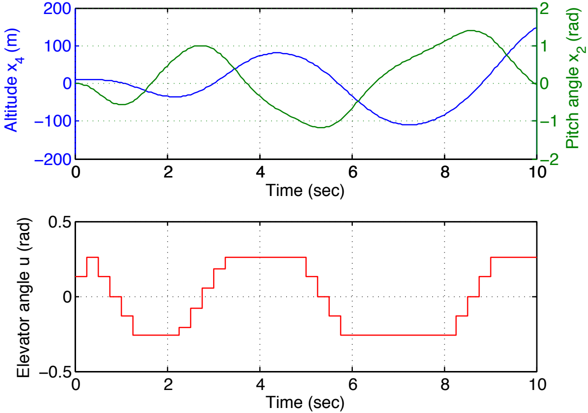
$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} {x_k}^T Q x_k + {u_k}^T R u_k$$
$$\text{s.t. } x_{k+1} = A x_k + B u_k$$
$$x_k \in \mathcal{X},\ u_k \in \mathcal{U}$$
$$x_0 = x(t)$$

Assume: $Q = Q^T \succeq 0,\ R = R^T \succ 0$

Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$



Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$

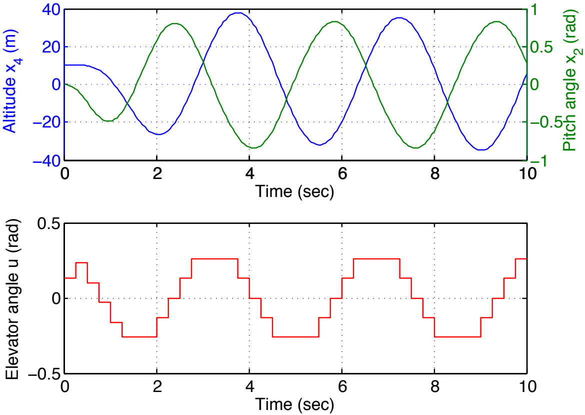
- Closed-loop system is unstable
- Applying LQR control and saturating the controller can lead to instability!

Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller uses the knowledge that the elevator will saturate, but it does not consider the rate constraints.

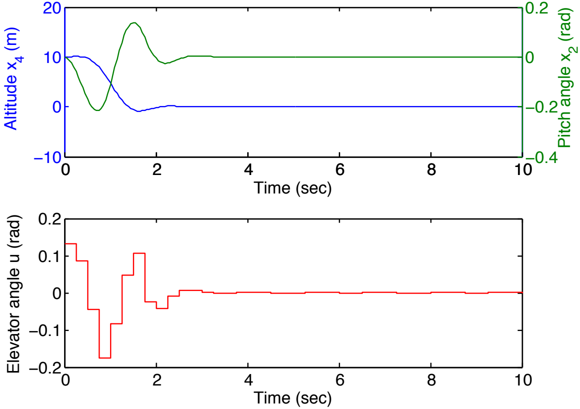
$\Rightarrow$ System does not converge to desired steady-state but to a limit cycle

Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$$



The MPC controller considers all constraints on the actuator

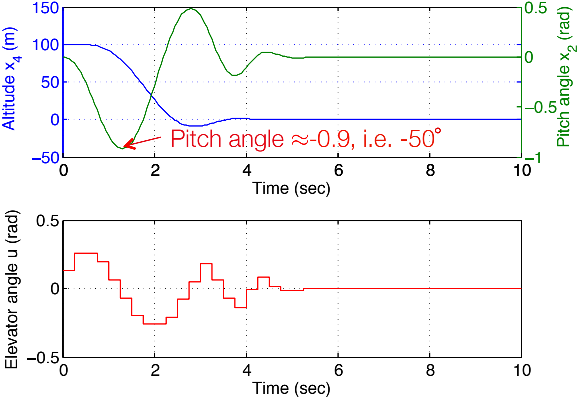
- Closed-loop system is stable
- Efficient use of the control authority

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



Increase step:
At time $t = 0$ the plane is flying with a deviation of 100m of the desired altitude, i.e. $x_0 = [0; 0; 0; 100]$

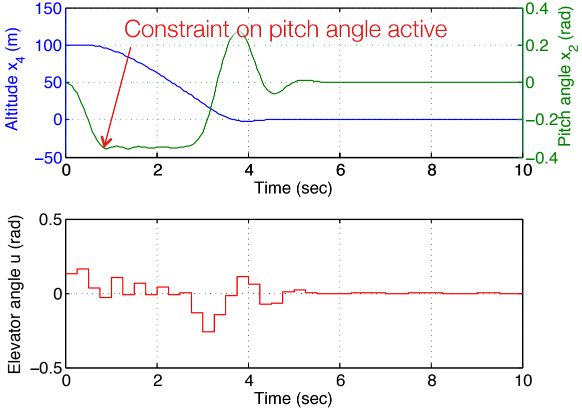
- Pitch angle too large during transient

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$, $N = 10$



Add state constraints for passenger comfort:

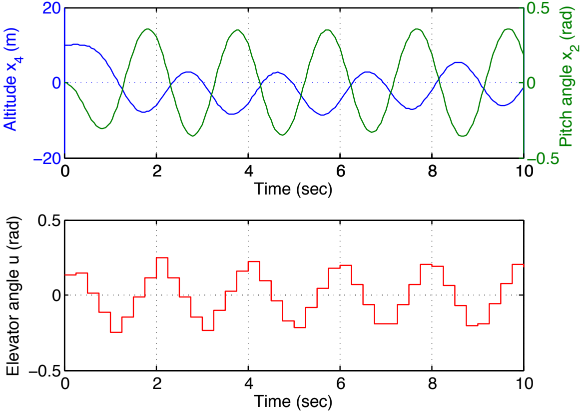
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

Loss of Feasibility and Stability

What can go wrong with "standard" MPC?

- No feasibility guarantee, i.e., the MPC problem may not have a solution
- No stability guarantee, i.e., trajectories may not converge to the origin

Summary: Feasibility and Stability

- Infinite-Horizon
  If we solve the RHC problem for $N = \infty$ (as done for LQR), then the open loop trajectories are the same as the closed loop trajectories. Hence
  - If problem is feasible, the closed loop trajectories will be always feasible
  - If the cost is finite, then states and inputs will converge asymptotically to the origin

- Finite-Horizon
  RHC is "short-sighted" strategy approximating infinite horizon controller. But

  - **Feasibility**. After some steps the finite horizon optimal control problem may become infeasible. (Infeasibility occurs without disturbances and model mismatch!)
  - **Stability**. The generated control inputs may not lead to trajectories that converge to the origin.

Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J_0^*(x_0) = \quad & \min_{U_0} & & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) & & \text{Terminal Cost} \\
& \text{subj. to} & & & & \\
& & & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 & & \\
& & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 & & \\
& & & x_N \in \mathcal{X}_f & & \text{Terminal Constraint} \\
& & & x_0 = x(t) & &
\end{aligned}
$$

$p(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

Choice of Terminal Set and Cost: Summary

- Terminal constraint provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attaction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

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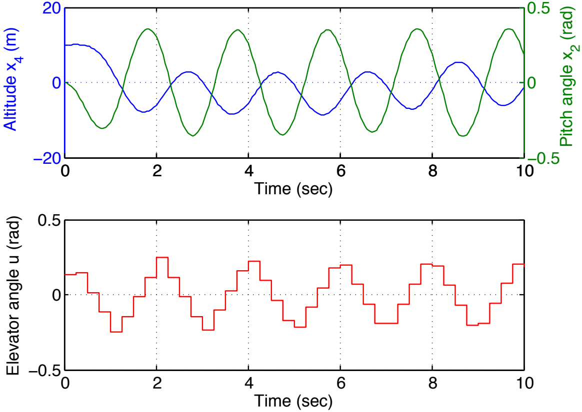

Decrease in the prediction horizon causes loss of the stability properties

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$
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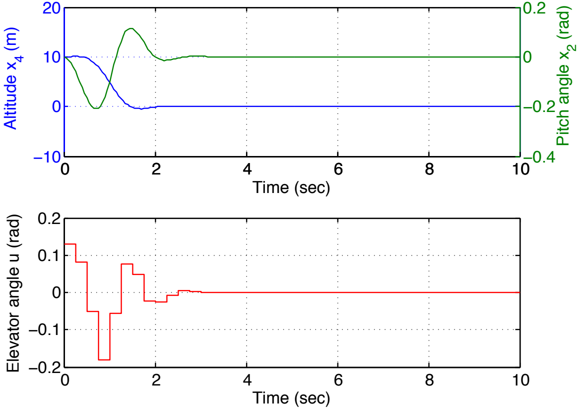
Problem parameters:

Sampling time 0.25sec,
$Q = I$, $R = 10$, $N = 4$



Inclusion of terminal cost and constraint provides stability

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon cost can be expressed in closed-form.
- These ideas extend to non-linear systems, but the sets are difficult to compute.

Extension to Nonlinear MPC

Consider the nonlinear system dynamics: $x(t+1) = g(x(t), u(t))$

$$
\begin{array}{rll}
J_0^*(x(t)) = & \min\limits_{U_0} & p(x_N) + \sum\limits_{k=0}^{N-1} q(x_k, u_k) \\
& \text{subj. to} & x_{k+1} = g(x_k, u_k),\ \ k = 0, \ldots, N-1 \\
& & x_k \in \mathcal{X},\ \ u_k \in \mathcal{U},\ \ k = 0, \ldots, N-1 \\
& & x_N \in \mathcal{X}_f \\
& & x_0 = x(t)
\end{array}
$$

- Presented assumptions on the terminal set and cost did not rely on linearity
- Lyapunov stability is a general framework to analyze stability of nonlinear dynamic systems

$\rightarrow$ Results can be directly extended to nonlinear systems.

However, computing the sets $\mathcal{X}_f$ and function p can be very difficult!

MPC: Tracking, Soft Constraints, Move-Blocking

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Spring Semester 2014
revised 29.04.2014

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Tracking problem

Consider the linear system model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

Goal: Track given reference r such that $y_k \to r$ as $k \to \infty$.

Determine the steady state target condition x_s, u_s:

$$\begin{aligned} x_s &= Ax_s + Bu_s \\ Cx_s &= r \end{aligned} \qquad \Longleftrightarrow \qquad \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

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Steady-state target problem

- In the presence of constraints: (x_s, u_s) has to satisfy state and input constraints.
- In case of multiple feasible u_s, compute 'cheapest' steady-state (x_s, u_s) corresponding to reference r:

$$\begin{aligned} \min \ & u_s^T R_s u_s \\ \text{s.t.} \ & \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \\ & x_s \in \mathcal{X}, \quad u_s \in \mathcal{U}. \end{aligned}$$

- In general, we assume that the target problem is feasible
- If no solution exists: compute reachable set point that is 'closest' to r:

$$\begin{aligned} \min \ & (Cx_s - r)^T Q_s (Cx_s - r) \\ \text{s.t.} \ & x_s = Ax_s + Bu_s \\ & x_s \in \mathcal{X}, \quad u_s \in \mathcal{U}. \end{aligned}$$

RHC Reference Tracking

We now use control (MPC) to bring the system to a desired steady-state condition (x_s, u_s) yielding the desired output $y_k \to r$.

The MPC is designed as follows

$$\begin{array}{ll} \min\limits_{u_0,\dots,u_{N-1}} & \|y_N - Cx_s\|_P^2 + \sum\limits_{k=0}^{N-1} \|y_k - Cx_s\|_Q^2 + \|u_k - u_s\|_R^2 \\ \text{subj. to} & \text{model} \\ & \text{constraints} \\ & x_0 = x(t). \end{array}$$

Drawback: controller will show **offset** in case of unknown model error or disturbances.

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Soft Constraints: Motivation

- Input constraints are dictated by physical constraints on the actuators and are usually “hard”
- State/output constraints arise from practical restrictions on the allowed operating range and are **rarely hard**
- Hard state/output constraints always lead to *complications in the controller implementation*
  - Feasible operating regime is constrained even for stable systems
  - Controller patches must be implemented to generate reasonable control action when measured/estimated states move outside feasible range because of disturbances or noise
- In industrial implementations, typically, state constraints are **softened**

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Mathematical Formulation

- Original problem:

$$\begin{array}{ll} \min\limits_{z} & f(z) \\ \text{subj. to} & g(z) \leq 0 \end{array}$$

  Assume for now $g(z)$ is scalar valued.

- "Softened" problem:

$$\begin{array}{ll} \min\limits_{z,\epsilon} & f(z) + l(\epsilon) \\ \text{subj. to} & g(z) \leq \epsilon \\ & \epsilon \geq 0 \end{array}$$

Requirement on $l(\epsilon)$

If the original problem has a feasible solution z^*, then the softened problem should have the same solution z^*, and $\epsilon = 0$.

Note: $l(\epsilon) = v \cdot \epsilon^2$ does not meet this requirement for any $v > 0$ as demonstrated next.

Main Result

Theorem (Exact Penalty Function)

$l(\epsilon) = u \cdot \epsilon$ satisfies the requirement for any $u > u^* \geq 0$, where u^* is the optimal Lagrange multiplier for the original problem.

- **Disadvantage:** $l(\epsilon) = u \cdot \epsilon$ renders the cost non-smooth.
- Therefore in practice, to get a smooth penalty, we use

$$l(\epsilon) = u \cdot \epsilon + v \cdot \epsilon^2$$

  with $u > u^*$ and $v > 0$.
- Extension to multiple constraints $g_j(z) \leq 0,\ j = 1, \ldots, r$:

$$l(\epsilon) = \sum_{j=1}^{r} u_j \cdot \epsilon_j + v_j \cdot \epsilon_j^2 \qquad (1)$$

  where $u_j > u_j^*$ and $v_j > 0$ can be used to weight violations (if necessary) differently.

Explicit Model Predictive Control

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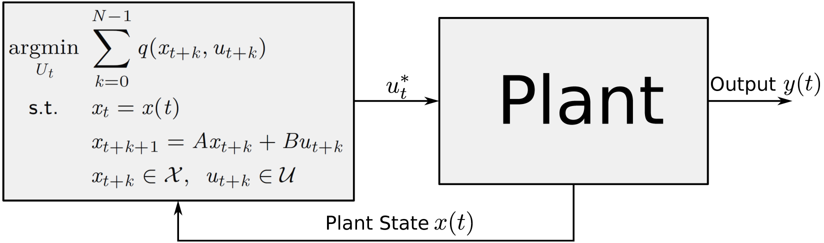
Spring Semester 2014
revised 29.04.2014

Introduction



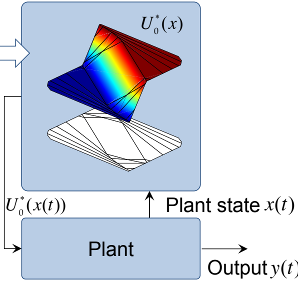
- Requires at each time step on-line solution of an optimization problem

Introduction

OFFLINE

$$U_0^*(x(t)) = \text{argmin} \;\; x_N^T P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$$
$$\text{subj. to} \;\; x_0 = x(t)$$
$$x_{k+1} = A x_k + B u_k, \;\; k = 0, \ldots, N-1$$
$$x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \;\; k = 0, \ldots, N-1$$
$$x_N \in \mathcal{X}_f$$

ONLINE



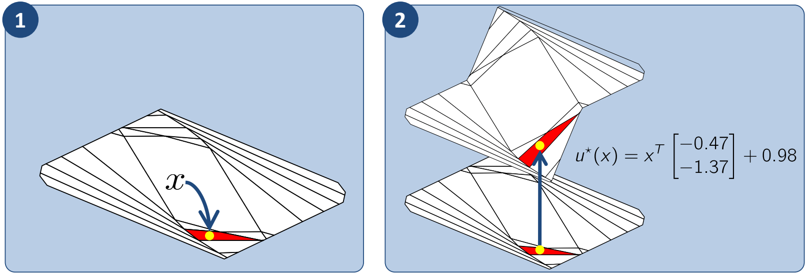
- Optimization problem is parameterized by state
- Pre-compute control law as function of state x
- Control law is piecewise affine for linear system/constraints

Result: Online computation dramatically reduced and *real-time*
Tool: *Parametric programming*

Online evaluation: Point location

Calculation of piecewise affine function:

1. Point location
2. Evaluation of affine function



Real-time MPC Software Toolbox

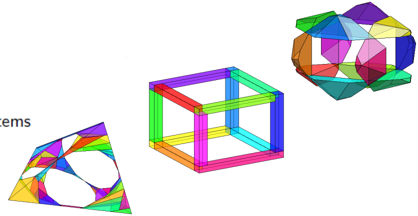
Formal specification
- YALMIP
- HYSDEL
- Linear + Hybrid models

Control law
- Explicit MPC
- Fixed-complexity solutions

Software synthesis
- Real-time workshop
- Bounded-time solvers
- Verifiable code generation

Verified controller

Multi-Parametric Toolbox (MPT)
- Computational geometry
- Multi-parametric programming
- Control of linear and hybrid systems



Hybrid Model Predictive Control

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Spring Semester 2014

Introduction

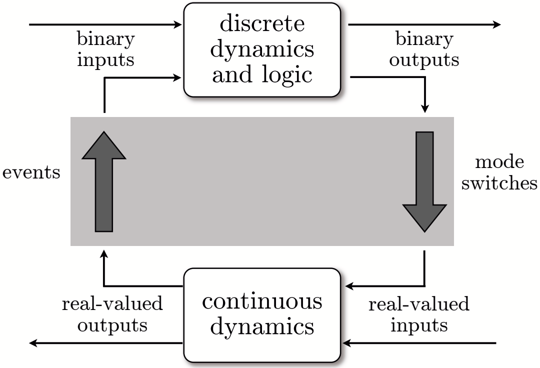
Up to this point: Discrete-time linear systems with linear constraints.

We now consider MPC for systems with

1. **Continuous dynamics**: described by one or more difference (or differential) equations; states are continuous-valued.
2. **Discrete events**: state variables assume *discrete* values, e.g.
   - binary digits $\{0, 1\}$,
   - $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\ldots$
   - finite set of symbols

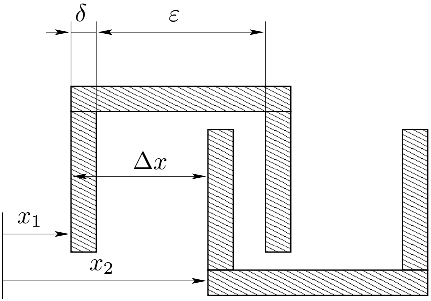
Hybrid systems: Dynamical systems whose state evolution depends on an interaction between continuous dynamics and discrete events.

Introduction



Hybrid systems: Logic-based discrete dynamics and continuous dynamics interact through events and mode switches

Mechanical System with Backlash

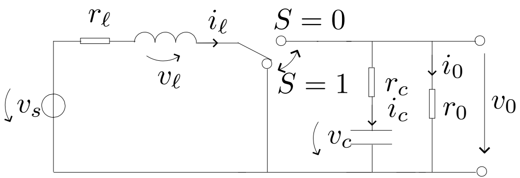


- **Continuous dynamics**: states x_1, x_2, $\dot{x}_1$, $\dot{x}_2$.
- **Discrete events**:
  a) *"contact mode"* $\Rightarrow$ mechanical parts are in contact and the force is transmitted. Condition:

$$[(\Delta x = \delta) \wedge (\dot{x}_1 > \dot{x}_2)] \bigvee [(\Delta x = \varepsilon) \wedge (\dot{x}_2 > \dot{x}_1)]$$

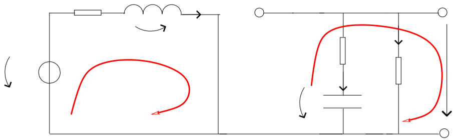
  b) *"backlash mode"* $\Rightarrow$ mechanical parts are not in contact

DCDC Converter

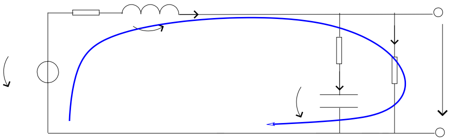


- **Continuous dynamics**: states v_ℓ, i_ℓ, v_c, i_c, v_0, i_0
- **Discrete events**: $S = 0$, $S = 1$

Mode 1 ($S = 1$)



Mode 2 ($S = 0$)



Mixed Logical Dynamical Systems

Goal: Describe hybrid system in form compatible with optimization software:

- continuous and boolean variables
- linear equalities and inequalities

Idea: associate to each Boolean variable p_i a binary integer variable δ_i:

$$p_i \Leftrightarrow \{\delta_i = 1\}, \quad \neg p_i \Leftrightarrow \{\delta_i = 0\}$$

and embed them into a set of constraints as **linear integer inequalities**.

Two main steps:

1. Translation of Logic Rules into Linear Integer Inequalities
2. Translation continuous and logical components into Linear Mixed-Integer Relations

Final result: a compact model with linear equalities and inequalities involving real and binary variables

MLD Hybrid Model

A DHA can be converted into the following MLD model

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 u_t + B_2 \delta_t + B_3 z_t \\ y_t &= Cx_t + D_1 u_t + D_2 \delta_t + D_3 z_t \\ E_2 \delta_t + E_3 z_t &\leq E_4 x_t + E_1 u_t + E_5 \end{aligned}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_\ell}$, $\delta \in \{0,1\}^{r_\ell}$ and $z \in \mathbb{R}^{r_c}$.

Physical constraints on continuous variables:

$$\mathcal{C} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathbb{R}^{n_c+m_c} \;\middle|\; Fx_c + Gu_c \leq H \right\}$$

HYbrid System DEscription Language

HYSDEL

- based on DHA
- enables description of discrete-time hybrid systems in a compact way:
  - automata and propositional logic
  - continuous dynamics
  - A/D and D/A conversion
  - definition of constraints
- automatically **generates MLD models** for MATLAB
- freely available from:

`http://control.ee.ethz.ch/~hybrid/hysdel/`

Optimal Control for Hybrid Systems: General Formulation

Consider the CFTOC problem:

$$J^*(x(t)) = \min_{U_0} \; p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k, \delta_k, z_k),$$

$$\text{s.t.} \left\{ \begin{array}{l} x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\ \\ E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\ \\ x_N \in \mathcal{X}_f \\ x_0 = x(t) \end{array} \right.$$

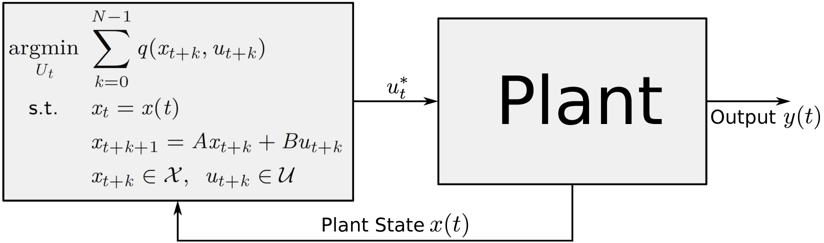
where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b}$, $\delta \in \{0,1\}^{r_b}$ and $z \in \mathbb{R}^{r_c}$ and

$$U_0 = \{u_0, u_1, \ldots, u_{N-1}\}$$

Mixed Integer Optimization

Model Predictive Control of Hybrid Systems

MPC solution: Optimization in the loop



As for linear MPC, at each sample time:

- Measure / estimate current state $x(t)$
- Find the optimal input sequence for the entire planning window N:
  $U_t^* = \{u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*\}$
- Implement only the *first* control action u_t^*
- **Key difference: Requires online solution of an MILP or MIQP**