Insights into Large Complex Systems via Random Matrix Theory

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Random Matrix Theory

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- milestones

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- **applications**: biology, data sciences, economics, information theory, machine learning, wireless communications,. . .

Product of Random Matrices

joint works with Akemann, Hero, Kieburg, Liu, Tarokh, Zhang, Zheng

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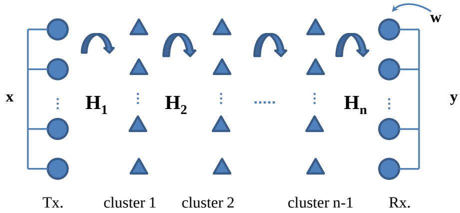

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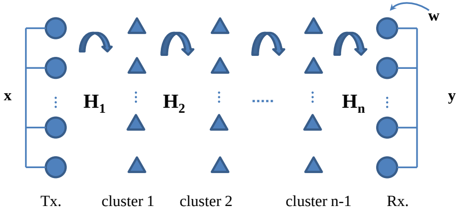

- an earlier attempt via eigenvalues and singular values relation*

* W., Zheng, Tirkkonen, Hämäläinen [2013] On the ergodic mutual information of multiple cluster scattering MIMO channels, *IEEE Commun. Lett.*

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$$f_j(x) = G_{0,m}^{m,0}\left(x \,\middle|\, \begin{matrix} - \\ 0, \ldots, 0, j-1 \end{matrix}\right) = \frac{1}{2\pi\imath} \oint_{\mathcal{L}} \mathrm{d}u \, x^{-u} \Gamma^{m-1}(u) \Gamma(u+j-1)$$

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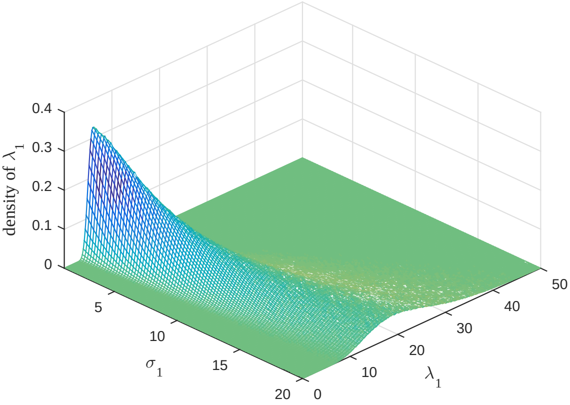

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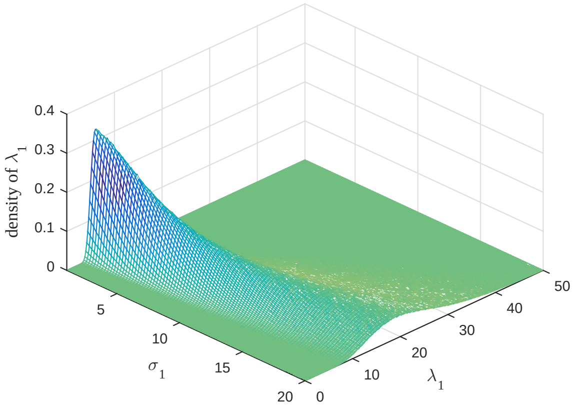

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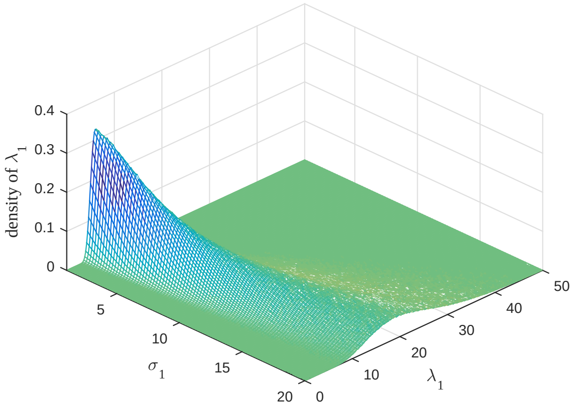

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Applications to Other Large Complex Systems

Signal Processing

joint works with Dharmawansa, Liang, McKay, Tirkkonen

Signal Detection

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- $\mathbf{R} = \mathbf{Y}\mathbf{Y}^\dagger$ data sample covariance; $\mathbf{E}$ noise sample covariance

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- $p > 1$ but unknown*

$$T_{\mathsf{ST}} = \frac{\det(\mathbf{R})}{\left(\frac{1}{m}\operatorname{tr}(\mathbf{R})\right)^{m}}$$

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‡ **W.**, Tirkkonen, Liang [2014] Multi-source signal detection with arbitrary noise covariance, *IEEE Trans. Signal Process.*

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joint works with Corander, Pitaval, Tirkkonen

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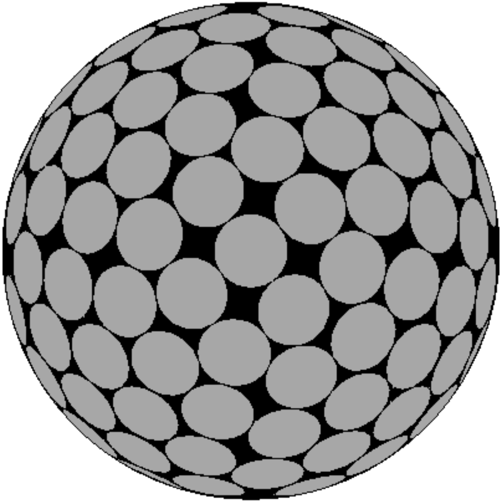
- volume of metric ball, $\quad \mu\left(B\left(r\right)\right) = \int_{d\left(\mathbf{G}, \mathbf{G}'\right) \leq r} f\left(\mathcal{G}\right) \mathrm{d}\mathcal{G}$

Volume of Metric Balls



Unitary Group

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* **W.**, Pitaval, Corander, Tirkkonen From random matrix theory to coding theory: Volume of a metric ball in unitary group, *IEEE Trans. Inf. Theory*, submitted, arXiv:1506.07259

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Grassmann Manifold

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Large Complex Systems

Wireless Communications

Signal Processing

Random Matrix Theory

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Information Theory

Statistical Physics

Data Sciences

Machine Learning